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Lesson 21: Comparing Linear and Exponential Functions Again

Student Outcomes

* Students create models and understand the differences between linear and exponential models that are represented in different ways.

Lesson Notes

Students have spent several lessons in this module studying linear and exponential functions. This lesson has students compare and contrast the two functions to solidify their understanding of each. Students then learn how to differentiate between the two functions by recognizing a constant rate of change or a common quotient in data tables, or by recognizing the features in their graphs or formulas.

Classwork

Opening Exercise (10 minutes)

This table format will allow students to compare linear and exponential models in several ways. This exercise reminds students what linear and exponential models are and allows them to easily see the difference between them. You can either give students the example or have them write their own for each model. Once students have finished, have them discuss their work in pairs or in groups. Tie this together with a class discussion, paying particular attention to the different wording students use and the real-world examples that they give. In a way similar to the prior lesson, students will use MP.7 to identify the meaning of structural components of each function type.

Opening Exercise

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|  | Linear Model | Exponential Model |
| General Form |  |  |
| Meaning of parameters and | is the slope of the line, or the constant rate of change,  is the -intercept, or the value at . | is the -intercept, or the value when ,  is the base, or the constant quotient of change. |
| Example |  |  |
| Rule for finding from | Starting at , to find , add to . | Starting at , to find , multiply by . |
| Table | |  |  | | --- | --- | |  |  | |  |  | |  |  | |  |  | |  |  | | |  |  | | --- | --- | |  |  | |  |  | |  |  | |  |  | |  |  | |
| Graph |  |  |
| Story Problem Example | Charles had in his bank account and earned per week as an allowance. | There were goldfish in the tank and the population doubled every month. |

Exercises 1–2 (12 minutes)

*Scaffolding:*

If students are struggling with the type of function, have them plot the points.

In this exercise, students complete the table for the function and find a progression of differences of two outputs of for two inputs that have a difference of 1 unit (i.e., they find the change in , , where the difference between and is 1.) If is constant for all input values and in the table, students identify the function given by the table as *potentially* linear. Similarly, when the quotient, , is a constant for all input values and , they identify the function as *potentially* exponential. This activity can be done individually or in groups. Upon completion, a class discussion should take place.

Be sure that students understand the following telltale signs for recognizing linear and exponential functions from a table of input-output pairs: for each , consider all inputs that have a difference of units,

* if the **difference** between their corresponding outputs is always the same constant, then the input-output pairs in the table can be modeled by a **linear** function;
* if the **quotient** between their corresponding outputs is always the same constant, then the input-output pairs in the table can be modeled by an **exponential** function.

In the exercise below, we use , but you can point out to students that these statements also make sense for , , etc.

Exercises 1–2

1. For each table below, assume the function is defined for all real numbers. Calculate in the last column in the tables below and show your work (the symbol in this context means “change in”). What do you notice about ? Could the function be linear or exponential? Write a linear or exponential function formula that generates the same input-output pairs as given in the table.

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Students should see that is constant for any two inputs that have a difference of unit, which implies that the function could be a linear function.

To the teacher: Point out that there are infinitely many functions that map each input in the table to the corresponding output, but only one of them is linear and none of them can be exponential.

In this example (and other linear examples like it), the constant change in depends on the choice of . However, the rate of change in per the distance is always the constant , that is: for all and . This constant rate of change is the slope of the line.

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If the entries in the table were considered as a geometric sequence, then the common quotient would be . ***Since , . Since , we must have or . Hence, .***

In this table, students should see that is not constant for any two inputs that have a difference of unit, which implies that the function cannot be a linear function. However, there is a common quotient between inputs that have a difference of 1 unit: . Hence the function could be exponential.

To the teacher: Again, point out that there are infinitely many functions that map each input in the table to the corresponding output, but none of them can be linear and only one of them can be of the form (exponential).

1. Terence looked down the second column of the table below and noticed that . Because of his observation, he claimed that the input-output pairs in this table could be modeled with an exponential function. Explain why Terence is correct or incorrect. If he is correct, write a formula for the exponential function that generates the input-output pairs given in the table. If he is incorrect, determine and write a formula for a function that generates the input-output pairs given in the table.

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Terence is incorrect. This is because the distance between consecutive -values is not 1 (not constant).

As you walk around the classroom, some of your students may claim he is correct and use the first two input-output pairs to get the formula . Ask them to see if it satisfies the remaining input-output pairs. You can guide (hint) to your students what function models this table by asking them to compute the average rate of change (you may need to recall this term for them) between input-output pairs. If they do, they will quickly see that the average rate of change is always . They should then be able to quickly derive the linear function using the first two input-out pairs. Encourage them to check their answers.

In groups, have students look at the Exercises 3 and 4. After groups have had time to discuss their solutions, have them discuss their results as a class. The purpose of these exercises is to help students understand when a situation models exponential or linear growth or decay. Exercise 4 re-emphasizes that the value of an exponential function with a base greater than 1 will always exceed any linear function for large enough positive input numbers. Exercise 4 highlights MP.3 as it calls on students to evaluate a hypothetical claim.

Exercise 3 (14 minutes)

Exercise 3

1. A river has an initial minnow population of that is growing at per year. Due to environmental conditions, the amount of algae that minnows use for food is decreasing, supporting fewer minnows each year. Currently, there is enough algae to support minnows. Is the minnow population increasing linearly or exponentially? Is the amount of algae decreasing at a linear or exponential rate? In what year will the minnow population exceed the amount of algae available?

The minnow population is increasing exponentially. The amount of algae is decreasing the food supply that can be used to support minnows linearly. Solution methods to finding when the minnow population will exceed the amount of algae may vary. Here’s one solution: We need to find the number of years such that (why?). For , we get is false, but for , we get that is true. Hence, some time after the third year the minnow population will exceed the amount of algae available.

1. Using a calculator, Joanna made the following table and then made the following conjecture: is always greater than . Is Joanna correct? Explain.

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The first function is exponential and the second is linear, so the exponential will eventually exceed the linear. (Alternatively, students could extend the table to larger values of to show that the exponential function will eventually exceed the linear function.) , but .

Closing (4 minutes)

* How can you tell whether input-output pairs in a table are describing a linear relationship or an exponential relationship?

Lesson Summary

* Suppose that the input-output pairs of a bivariate dataset have the following the property: for every two inputs that are a given difference apart, the difference in their corresponding outputs is constant. Then an appropriate model for that dataset could be a linear function.
* Suppose that the input-output pairs of a bivariate dataset have the following the property: for every two inputs that are a given difference apart, the quotient of their corresponding outputs is constant. Then an appropriate model for that dataset could be an exponential function.
* An increasing exponential function will eventually exceed any linear function. That is, if is an exponential function with and , and is any linear function, then there is a real number such that for all , then . Sometimes this is not apparent in a graph displayed on a graphing calculator; that is because the graphing window does not show enough of the graph to show the sharp rise of the exponential function in contrast with the linear function.

Exit Ticket (5 minutes)

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Lesson 21: Comparing Linear and Exponential Functions Again

Exit Ticket

Here is a classic riddle: Mr. Smith has an apple orchard. He hires his daughter, Lucy, to pick apples and offers her two payment options.

Option A: per bushel of apples picked.

Option B: for picking one bushel, for picking two bushels, for picking three bushels, and so on, with the amount of money tripling for each additional bushel picked.

1. Write a function to model each option.
2. If Lucy picks bushels of apples, which option should she choose?
3. If Lucy picks bushels of apples, which option should she choose?
4. How many bushels of apples does Lucy need to pick to make option B better for her than option A?

Exit Ticket Sample Solutions

Here is a classic riddle: Mr. Smith has an apple orchard. He hires his daughter, Lucy, to pick apples and offers her two payment options.

Option A: per bushel of apples picked.

Option B: for picking one bushel, for picking two bushels, for picking three bushels, and so on, with the amount of money tripling for each additional bushel picked.

1. Write a function to model each option.

Option A: where is the number of bushels of apples picked.

Option B: where is the number of bushels of apples picked.

1. If Lucy picks bushels of apples, which option should she choose?

Option A: Option B: Option A is better.

1. If Lucy picks bushels of apples, which option should she choose?

Option A: Option B: Option B is much better.

1. How many bushels of apples does Lucy pick to make option B better for her than option A?

The eighth bushel picked is when the exponential function exceeds the linear function.

Problem Set Sample Solutions

For each table in Problems 1–6, classify the data as describing a linear relationship, an exponential growth relationship, an exponential decay relationship, or neither. If the relationship is linear, calculate the constant rate of change (slope), and write a formula for the linear function that models the data. If the function is exponential, calculate the common quotient for input values that are distance 1 apart, and write the formula for the exponential function that models the data. For each linear or exponential function found, graph the equation .

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Common quotient: , exponential decay, .



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Constant rate of change: , linear, .



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Ratio of change: , exponential growth,

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Rate of change: , linear,



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Neither.

1. Here is a variation on a classic riddle: Jayden has a dog-walking business. He has two plans. Plan 1 includes walking a dog once a day for a rate of $5/day. Plan 2 also includes one walk a day, but charges 1 cent for 1 day, 2 cents for 2 days, 4 cents for 3 days, 8 cents for 4 days, and continues to double for each additional day. Mrs. Maroney needs Jayden to walk her dog every day for two weeks. Which plan should she choose? Show the work to justify your answer.

Plan #1: . She will pay for all two weeks under Plan 1.

Plan #2: The price can be modeled using the function , which describes the amount she pays for days. Then,

or .

She will pay under Plan 2 for two weeks.

Plan 1 is the better choice.

1. **Tim deposits money in a Certificate of Deposit account. The balance (in dollars) in his account years after making the deposit is given by for .** 
   1. Explain, in terms of the structure of the expression used to define , why Tim's balance can never be .

and positive powers of are larger than 1, thus the minimum value attains, if , is . In the context given, a CD account grows in value over time so with a deposit of the value will never drop to .

* 1. By what percent does the value of grow each year? Explain by writing a recursive formula for the sequence etc.

Writing out the sequence, we see:

Year 1:   
Year 2:   
Year 3:   
…

Thus, or , showing that grows by 6% a year.

* 1. By what percentages does the value of grow every two years? (Hint: Use your recursive formula to write in terms of .)

Since, , we can write,

or .

Hence, the amount in his account grows by every two years.

1. Your mathematics teacher asks you to sketch a graph of the exponential function for a number between and inclusively, using a scale of units to one inch for both the - and -axes.
   1. What are the dimensions in feet of the roll of paper you will need to sketch this graph?

The roll would need to be a bit wider than in. ( ft.) and, after rounding up, about in. (or ft.) long.

* 1. How many more feet of paper would you need to add to the roll in order to graph the function on the interval ?

It would require approximately more feet of paper. Teachers: look for the easy solution: .

* 1. Find an so that the linear function is greater than for all such that , but .

There are many possible answers: any number between (roughly) 5 and will do. Note that and , so any slope such that and will do. Since , solving , or might produced such a linear function. We need to check only that the function   
 satisfies , which it does: .